CONCENTRICITY IN 3-MANIFOLDS

BY

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1. **Introduction.** Two compact 3-manifolds M and N with nonempty boundaries and with $N \subset \operatorname{Int} M$ are said to be *concentric* if $\mathrm{Cl}(M-N)$ is homeomorphic to $(\operatorname{Bd} M) \times I$, where I is the closed unit interval $[0,1]$. In [10] it was shown that, if B, B_0, and B_1 are polyhedral solid tori of genus 1 in S^3 with $B_0 \subset \operatorname{Int} B \subset B \subset \operatorname{Int} B_1$, then B_0 and B_1 are concentric if and only if B is concentric with each of B_0 and B_1.

The principal theorems on concentricity proved in this paper are Theorems 1 and 2 in §3. In Theorem 2 it is shown that, if M_0 and M_1 are concentric compact 3-manifolds and N is a compact 3-manifold tamely imbedded in $\operatorname{Int} M_1$ and containing M_0 in its interior, with Bd N homeomorphic to Bd M_0, then N is concentric with (and hence homeomorphic with) each of M_0 and M_1.

Several applications of Theorems 1 and 2 are given in §4. For instance, in Theorem 3 it is shown that two compact 3-manifolds are homeomorphic if their interiors are homeomorphic. In Theorem 5 it is shown that, if M is a compact 3-manifold with nonempty boundary and with spine K, and if the 3-manifold X is the union of a sequence $\{h_n(M)\}_1^\infty$ of homeomorphic images of M such that $h_n(M) \subset \operatorname{Int} h_{n+1}(M)$ and $h_n(K) = h_{n+1}(K)$ for each n, then X is homeomorphic to $\operatorname{Int} M$.

An *n-manifold* M is a connected separable metric space each of whose points has a neighborhood whose closure is homeomorphic to the closed n-cell I^n. The *interior* $\operatorname{Int} M$ of M is the set of those points of M which have neighborhoods homeomorphic to Euclidean n-space E^n, while the *boundary* $\operatorname{Bd} M$ of M is defined by $\operatorname{Bd} M = M - \operatorname{Int} M$. A *closed* n-manifold is a compact n-manifold whose boundary is empty. All 3-manifolds mentioned are assumed to be triangulated [2], [3], [13].

2. **Preliminary lemmas.** This section is devoted to the proofs of the lemmas which will be needed for the proofs of Theorems 1 and 2. All sets mentioned in this section are assumed to be *polyhedral*, whether or not this is explicitly stated. The closed unit interval [0, 1] is denoted throughout by I.

LEMMA 1. *Let T be a torus (a closed orientable 2-manifold of genus 1)*

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and S a closed 2-manifold imbedded in $\text{Int}(T \times I)$ *and separating* $T \times 0$ *and* $T \times 1$ *in* $T \times I$. *Then S is not a 2-sphere.*

Proof. By sewing a solid torus (a 3-cell with one handle) onto $T \times I$ by identifying its boundary with $T \times 0$, we may consider $T \times 0$ and $T \times 1$ as the boundaries of two concentric solid tori B_0 and B_1 respectively with $B_0 \subset \text{Int}\, B_1$. Then B_0 and B_1 have a common center-line $c \subset \text{Int}\, B_0$. If S were a 2-sphere separating $\text{Bd}\, B_0$ and $\text{Bd}\, B_1$, then S would be the boundary of a 3-cell $K \subset \text{Int}\, B_1$ containing B_0 in its interior. But then c would be nullhomotopic in B_1, which is a contradiction. Hence S is not a 2-sphere.

LEMMA 2. *Let T be a torus with one hole* (*a compact orientable 2-manifold of genus 1 with one boundary component*) *and* $B = \text{Bd}\, T$. *Let* $S_1, \cdots, S_k$ *be mutually disjoint compact 2-manifolds such that* $\text{Int}\, S_i \subset \text{Int}(T \times I)$ *and* $\text{Bd}\, S_i \subset \text{Int}(B \times I)$ *for each* $i = 1, \cdots, k$. *If* $\bigcup_1^k S_i$ *separates* $T \times 0$ *'and* $T \times 1$ *in* $T \times I$, *then at least one of the* S_i *is of positive genus.*

Proof. Let D be a disk with boundary B so that $T \cup D$ is a torus T^*, and consider $T \times I$ as a subset of $T^* \times I$, with $T^* \times I$ imbedded in the 3-sphere S^3. Now let the boundary components of S_i be $r_{i1}, r_{i2}, \cdots, r_{in_i}$, $i = 1, \cdots, k$, and for each i and j let D_{ij} be a disk such that $\text{Bd}\, D_{ij} = r_{ij}$ and $\text{Int}\, D_{ij} \subset \text{Int}(D \times I)$, with all these disks D_{ij} mutually disjoint. If $S_i^* = S_i \cup \bigcup_{j=1}^{n_i} D_{ij}$, $i = 1, \cdots, k$, then $S_1^*, \cdots, S_k^*$ is a collection of mutually disjoint closed orientable 2-manifolds in $\text{Int}(T^* \times I)$ such that $\bigcup_1^k S_i^*$ separates $T^* \times 0$ and $T^* \times 1$ in $T^* \times I$. By the Phragmen-Brouwer property of S^3 [12, p. 359], some one S_p^* must separate $T^* \times 0$ and $T^* \times 1$. Lemma 1 then implies that S_p^*, and hence S_p, is of positive genus.

LEMMA 3. *Let P be a projective plane and let* $S_1, \cdots, S_k$ *be mutually disjoint closed 2-manifolds in* $\text{Int}(P \times I)$ *such that* $\bigcup_1^k S_i$ *separates* $P \times 0$ *and* $P \times 1$ *in* $P \times I$. *Then at least one of the* S_i *is nonorientable.*

Proof. Let M be the closure of the component of $(P \times I) - \bigcup_1^k S_i$ which contains $P \times 0$ but not $P \times 1$. If each of $S_1, \cdots, S_k$ is orientable, then M is a compact 3-manifold, one of whose boundary components is a projective plane, while the others are all orientable. Since the Euler characteristic of the projective plane is odd, while the Euler characteristic of any orientable closed 2-manifold is even, this is a contradiction, because the Euler characteristic of the boundary of any compact 3-manifold is even [15, p. 223]. Therefore one of the S_i must be nonorientable.

LEMMA 4. *Let M be a Moebius strip with* $B = \text{Bd}\, M$. *Let* $S_1, \cdots, S_k$ *be mutually disjoint compact 2-manifolds such that* $\text{Int}\, S_i \subset \text{Int}(M \times I)$ *and* $\text{Bd}\, S_i \subset \text{Int}(B \times I)$ *for each* $i = 1, \cdots, k$. *If* $\bigcup_1^k S_i$ *separates* $M \times 0$ *and* $M \times 1$ *in* $M \times I$, *then at least one of the* S_i *is nonorientable.*

Proof. This follows from Lemma 3 in the same way that Lemma 2 followed from Lemma 1.

LEMMA 5. *Let B and C be closed 2-manifolds with C imbedded in the interior of $B \times I$ and separating $B \times 0$ and $B \times 1$ in $B \times I$. Then C is orientable or not according as B is orientable or not, and the genus of C is equal to or greater than the genus of B.*

Proof. First suppose that B is orientable of positive genus n. Then $B \times I$ can be imbedded in S^3, and this implies that C is orientable, since S^3 does not contain a nonorientable closed 2-manifold. Let each of $T_1, T_2, \cdots, T_n$ be a torus with one hole and K a 2-sphere with n holes, such that $B = K \cup T_1 \cup \cdots \cup T_n$, with $K \cap T_i$ being the boundary curve A_i of T_i for each $i = 1, \cdots, n$, and $\operatorname{Bd} K = \bigcup_1^n A_i$. Then $C \cap (T_i \times I)$ must separate $T_i \times 0$ and $T_i \times 1$ in $T_i \times I$. If C is in general position with respect to $\bigcup_1^n A_i \times I$, then each component of $C \cap (T_i \times I)$ is a compact 2-manifold whose interior lies in $\operatorname{Int}(T_i \times I)$ and whose boundary lies in $\operatorname{Int}(A_i \times I)$. It then follows from Lemma 2 that some component S_i of $C \cap (T_i \times I)$ is of positive genus. Therefore, since the genus of C is at least as great as the sum of the genera of $S_1, \cdots, S_n$, it follows that the genus of C is at least n.

Now suppose that B is a nonorientable closed 2-manifold of genus n. In this case let each of $M_1, \cdots, M_n$ be a Moebius strip and K a 2-sphere with n holes, such that $B = K \cup M_1 \cup \cdots \cup M_n$, with $K \cap M_i$ being the boundary curve A_i of M_i for each $i = 1, \cdots, n$, and $\operatorname{Bd} K = \bigcup_1^n A_i$. It then follows from Lemma 4 that, for each i, at least one component S_i of $C \cap (M_i \times I)$ is nonorientable. Hence C is a nonorientable closed 2-manifold of genus at least n.

LEMMA 6. *Let B, C_1, and C_2 be closed 2-manifolds, with C_1 and C_2 disjoint and imbedded in* $\operatorname{Int}(B \times I)$ *in such a way that neither separates $B \times 0$ and $B \times 1$ in $B \times I$. Then $C_1 \cup C_2$ does not separate $B \times 0$ and $B \times 1$ in $B \times I$.*

Proof. Since the fundamental 2-cycle on $B \times 0$ generates the 2-dimensional homology group (mod 2 coefficients) of $B \times I$, the fundamental 2-cycle on C_i is either nullhomologous in $B \times I$ or is homologous to $B \times 0$. But in the latter case $C_i \cup (B \times 0)$ is the boundary of a compact 3-manifold in $B \times I$, so that C separates $B \times 0$ and $B \times 1$. Consequently C_i is the boundary of a compact 3-manifold M_i in $\operatorname{Int}(B \times I)$, $i = 1, 2$.

Now let A be a polygonal arc in $B \times I$ joining $B \times 0$ and $B \times 1$, not intersecting C_1 and piercing C_2 at each point of $A \cap C_2$ (either leaving or entering M_2). If L is a subarc of A contained in M_2 and with its endpoints on C_2, then we can replace L by an arc on C_2, and then piecewise-linearly deform the new A away from C_2, thus decreasing the number of components of $A \cap M_2$. After a finite number of alterations of this kind we obtain an arc from $B \times 0$ to $B \times 1$ which misses both C_1 and C_2. Therefore $C_1 \cup C_2$ does not separate $B \times 0$ and $B \times 1$ in $B \times I$.

LEMMA 7. *If the 2-complex K is the union of a disk D and a Moebius strip M such that $D \cap M$ is the center line of M, then K cannot be imbedded piecewise-linearly in any 3-manifold.*

Proof. Since K is collapsible, a regular neighborhood of it in any 3-manifold is a 3-cell [16]. It therefore suffices to notice that K cannot be imbedded piecewise-linearly in E^3, because the center-line of a polyhedral Moebius strip in E^3 links its boundary curve.

LEMMA 8. *Let B and C be homeomorphic closed 2-manifolds with C lying in the interior of $B \times I$ and separating $B \times 0$ and $B \times 1$ in $B \times I$. If m is a simple closed curve on C which is the boundary of a disk $D \subset \mathrm{Int}(B \times I)$ such that each component of $C \cap \mathrm{Int}\, D$ is a simple closed curve, then m is the boundary of a disk on C.*

Proof. Consider first the special case in which $\mathrm{Int} D$ does not intersect C. By virtue of Lemma 7, m is two-sided on C. Hence let m' be a simple closed curve on C close to m such that $m \cup m'$ is the boundary of a narrow annulus R on C, and let D' be a disk bounded by m' which lies close to and is disjoint with D. Since $C \cup D \cup D'$ separates $B \times 0$ and $B \times 1$ in $B \times I$, it follows that $(C - \mathrm{Int}\, R) \cup D \cup D'$ separates $B \times 0$ and $B \times 1$ in $B \times I$. For if L were an arc from $B \times 0$ to $B \times 1$ missing $(C - \mathrm{Int}\, R) \cup D \cup D'$, and hence intersecting $C \cup D \cup D'$ only in points of $\mathrm{Int}\, R$, then these intersections could be easily eliminated to obtain an arc from $B \times 0$ to $B \times 1$ missing $C \cup D \cup D'$.

Now if $\mathrm{Int} R$ does not separate C, then $(C - \mathrm{Int}\, R) \cup D \cup D'$ is a closed 2-manifold which has genus less than that of C and which separates $B \times 0$ and $B \times 1$ in $B \times I$. But this is a contradiction to Lemma 5, since the genera of B and C are equal.

Therefore $\mathrm{Int}\, R$ separates C into two components A and A' bounded by m and m' respectively. Since $A \cup D$ and $A' \cup D'$ are then disjoint closed 2-manifolds whose union $(C - \mathrm{Int}\, R) \cup D \cup D'$ separates $B \times 0$ and $B \times 1$ in R, it follows from Lemma 6 that at least one of them, say $A \cup D$, must separate $B \times 0$ and $B \times 1$ in $B \times I$. Hence Lemma 5 implies that the genus of $A \cup D$ is equal to that of C, so that $R \cup A'$ must be a disk on C whose boundary is m.

Now suppose that $\mathrm{Int}\, D$ intersects C in a finite positive number of simple closed curves, and let r be an interior one of these, i.e., one which bounds a disk E in D whose interior does not intersect C. Then by the special case considered above, r bounds a disk F on C. If $\mathrm{Int} F$ does not intersect D, we can eliminate the component r of $C \cap \mathrm{Int}\, D$ by replacing the subdisk E of D by F and then deforming $(D - E) \cup F$ away from C in a neighborhood of F. If $\mathrm{Int} F$ does intersect D, we can eliminate in the same way an interior (on F) component of $D \cap \mathrm{Int}\, F$.

After a finite number of alterations of this kind, we obtain a disk D' with $\mathrm{Bd}\, D' = m$ and with $C \cap \mathrm{Int}\, D'$ empty, so that the special case applies.

LEMMA 9. *Let N be a closed 2-manifold other than S^2, p a fixed point of N, and denote by L the polygonal arc $p \times I \subset N \times I$. Suppose that S is a polyhedral 2-sphere in $\mathrm{Int}(N \times I)$ such that $S \cap L$ is a finite set F containing more than two points, at each of which L pierces S. Then there is a simple closed polygon m in $S - F$ which separates F on S and is the boundary of a disk D such that $D \cap L = \varnothing$ and each component of $S \cap \mathrm{Int}\, D$ is a simple closed polygon.*

Proof. Since $N \neq S^2$, let C be a non-nullhomotopic simple closed polygon on N with $p \in C$. Denote by R the annulus $C \times I$. By general position it may be assumed that each component of $R \cap S$ is a simple closed polygon. If J is a component of $R \cap S$ which intersects L, then J must be nullhomotopic in R, since otherwise J would be homotopic to $C \times 0$, which is non-nullhomotopic in $N \times I$, whereas J is nullhomotopic in S. Hence $J \cap L$ consists of an even number of points, so that I contains an arc A which intersects L in precisely its endpoints a and b, with the union of A and the subarc ab of L being a simple closed curve which bounds a disk B in R. If the disk E is the regular neighborhood of A in a very fine subdivision of S, then $m = \mathrm{Bd}\, E$ is a simple closed polygon on S which separates $a \cup b$ and $F - (a \cup b)$, and is clearly the boundary of a disk D which misses L. The disk D will "fold" around L and will lie close to and on both sides of the disk B, with L piercing the 2-sphere $D \cup E$ only at a and b. Note that each component of $S \cap \mathrm{Int}\, D$ will be a simple closed polygon.

LEMMA 10. *Let each of A and B be a 2-sphere with n holes, and suppose that $r_1, \cdots, r_n$ and $s_1, \cdots, s_n$ are the collections of boundary components of A and B respectively. If $\mathrm{Int}\, B \subset \mathrm{Int}(A \times I)$ and $s_i = r_i \times \frac{1}{2}$ for each i, then there is a homeomorphism h of $A \times I$ onto itself such that $h \,|\, \mathrm{Bd}(A \times I)$ is the identity and $f(B) = A \times \frac{1}{2}$.*

Proof. The proof is by induction on n, the number of boundary components of A and B. If $n = 1$, i.e., if A and B are disks, then the closures C_0 and C_1 of the components of $(A \times I) - B$ containing $A \times 0$ and $A \times 1$ respectively are both 3-cells [1]. Defining $f(x) = x$ if $x \in \mathrm{Bd}(A \times I)$, f can be extended to take B homeomorphically onto $A \times \frac{1}{2}$. Since f now takes $\mathrm{Bd}\, C_0$ and $\mathrm{Bd}\, C_1$ homeomorphically onto $\mathrm{Bd}(A \times [0,\frac{1}{2}])$ and $\mathrm{Bd}(A \times [\frac{1}{2},1])$ respectively, it can be extended to take C_0 and C_1 homeomorphically onto $A \times [0,\frac{1}{2}]$ and $A \times [\frac{1}{2},1]$ respectively. f is now the desired homeomorphism.

If $n > 1$, let e be a polygonal arc which lies in $\mathrm{Int}\, A$ except for its endpoints $x \in r_1$ and $y \in r_2$, and denote by E the polyhedral disk $e \times I$. We will define a sequence of homeomorphisms of $A \times I$ onto itself, each the identity on $\mathrm{Bd}(A \times I)$, such that the image of B under their composition intersects E in an arc which, except for its endpoints $p = x \times \frac{1}{2}\varepsilon s_1$ and $q = y \times \frac{1}{2}\varepsilon s_2$, lies interior to B. However, to avoid proliferation of notation, each new image of B will still be denoted by B.

First adjust B so that it is in general position with respect to E. Then each of the components of $B \cap \operatorname{Int} E$ is of one of the following three types: (1) a simple closed curve, (2) an open arc with a single limit point, either p or q, (3) an open arc with limit points p and q.

Now let C be a minimal component of type 1, i.e., a simple closed curve of $B \cap \operatorname{Int} E$ which bounds a disk $D \subset E$ whose interior misses B. If C did not bound a disk in B, then C would be homotopic to a combination of the boundary curves of B, and therefore would not be nullhomotopic in $A \times I$. Hence C bounds a disk $D' \subset B$. The component C of $B \cap \operatorname{Int} E$ can then be eliminated by a homeomorphism which moves D' onto D and then deforms D slightly off E (see §7 of [5]). A finite number of steps of this kind suffice to eliminate all of the components of type 1.

Since the closure of each component of type 2 is a simple closed curve which bounds disks in both B and E, the components of type 2 can be removed in the same way as were those of type 1, except that all the deformations off E leave p and q fixed.

If there is more than one component of type 3, let C_1 and C_2 be the bottom two (the two which are nearest $A \times 0$ on E). Then the simple closed curve $p \cup C_1 \cup C_2 \cup q$ bounds a disk D on E and a disk D' on B. The components C_1 and C_2 of $B \cap \operatorname{Int} E$ are then removed by a homeomorphism which first moves D' onto D and then moves $D - (p \cup q)$ slightly off E. Since there is an odd number of components of type 3, and this alteration eliminates an even number of them, after a finite number of steps of this kind there remains only a single component of type 3, so that $B \cap E$ is then a single arc.

Now let e' be a second polygonal arc which lies in $\operatorname{Int} A$ except for its endpoints $x' \in r_1$ and $y' \in r_2$, such that e and e' are disjoint and $e \cup e'$ separates A into two components whose closures A_1 and A_2 are a disk and a sphere with $n-1$ holes respectively. Since $B \cap E$ is a single arc, we may choose e' so close to e that B also intersects $E' = e' \times I$ in a single arc. By a homeomorphism of $A \times I$ onto itself which is the identity on $\operatorname{Bd}(A \times I)$, we may move the arcs $B \cap E$ and $B \cap E'$ onto $e \times \frac{1}{2}$ and $e' \times \frac{1}{2}$ respectively. Now if $B_i = B \cap (A_i \times I)$, $i = 1,2$, then B_1 and B_2 are a disk and a sphere with $n-1$ holes respectively, such that $\operatorname{Int} B_i \subset \operatorname{Int}(A_i \times I)$ and $\operatorname{Bd} B_i = (\operatorname{Bd} A_i) \times \frac{1}{2}$, $i = 1,2$. It therefore follows by induction that there is a homoemorphism f_i of $A_i \times I$ onto itself such that $f_i | \operatorname{Bd}(A_i \times I)$ is the identity and $f_i(B_i) = A_i \times \frac{1}{2}$, $i = 1,2$. Since each of f_1 and f_2 is the identity on $(A_1 \times I) \cap (A_2 \times I)$, they combine to give the desired homeomorphism of $A \times I$ onto itself.

Lemma 11. *Let each of A and B be a torus with one hole, and let r and s be the boundary curves of A and B respectively. If* $\operatorname{Int} B \subset \operatorname{Int}(A \times I)$ *and*

$s = r \times \frac{1}{2}$, with B separating $A \times 0$ and $A \times 1$ in $A \times I$, then there is a homeomorphism h of $A \times I$ onto itself such that $h|\mathrm{Bd}(A \times I)$ is the identity and $f(B) = A \times \frac{1}{2}$.

Proof. Fill in the hole of A with a disk D to obtain a torus A^*, and let $B^* = B \cup (D \times \frac{1}{2})$. Now imbed $A^* \times I$ piecewise-linearly in S^3 so that the closure of each complementary domain is a solid torus. Then the closure of at least one of the complementary domains of B^* will also be a solid torus [1]. So we may assume that $A^* \times 0$, B^*, and $A^* \times 1$ are the boundaries of three solid tori T_0, T, and T_1, with T_0 and T_1 concentric, such that $T_0 \subset \mathrm{Int}\, T \subset T \subset \mathrm{Int}\, T_1$. Then the order of T_1 with respect to T is unity [14, p. 175]. If M is a meridianal disk of T_1 missing $D \times I$ and such that $M \cap T_0$ is a meridianal disk of T_0, T can therefore be adjusted (leaving $D \times I$ fixed) so that $M \cap T$ is a meridianal disk of T [14, p. 174]. Thus , if m is a meridianal curve on A, we may assume that $m \times I$ intersects B in a meridianal curve on B. Now let n be another meridianal curve of A close to m, with $m \cup n$ separating A into two components whose closures A_1 and A_2 are an annulus and a 2-sphere with three holes. Then $(m \times I) \cup (n \times I)$ will similarly separate B into an annulus B_1 and a 2-sphere with three holes B_2, such that $\mathrm{Int}\, B_i \subset \mathrm{Int}(A_i \times I)$ and $\mathrm{Bd}\, B_i \subset \mathrm{Int}((\mathrm{Bd}\, A_i) \times I)$. After moving the simple closed curves $B \cap (m \times I)$ and $B \cap (n \times I)$ onto $m \times \frac{1}{2}$ and $n \times \frac{1}{2}$ respectively, we then obtain the desired homeomorphism of $A \times I$ onto itself by applying Lemma 10 to $A_1 \times I$ and $A_2 \times I$ separately.

LEMMA 12. *Let M and N be Moebius strips with boundaries r and s respectively, such that $\mathrm{Int}\, N \subset \mathrm{Int}(M \times I)$ and $s = r \times \frac{1}{2}$, with N separating $M \times 0$ and $M \times 1$ in $M \times I$. Then there is a homeomorphism h of $M \times I$ onto itself such that $h|\mathrm{Bd}(M \times I)$ is the identity and $h(N) = M \times \frac{1}{2}$.*

Proof. The proof of this lemma follows closely the plan of the proof of Lemma 10. That is, starting with two arcs e and f spanning the boundary of M and separating M into two components, the closure of each of which is a disk, we eliminate the unnecessary components of the intersections of $E = e \times I$ and $F = f \times I$ with N. The only difference here lies in showing that, if C is a simple closed curve in $E \cap N$ which bounds a disk in E, then C also bounds a disk on N. But if C were not nullhomotopic on N, then C would be homotopic to either the centerline or the boundary curve of N, neither of which is nullhomotopic in $M \times I$. The remainder of the proof is the same as that of Lemma 10, and is therefore omitted.

LEMMA 13. *Let C_1 and C_2 be two non-nullhomotopic simple closed polygons on a projective plane P. Then C_1 and C_2 must intersect.*

Proof. Suppose, to the contrary, that $C_1 \cap C_2$ is empty. Neither C_1 nor C_2 alone separates P. For if C_1 separated P, then each component of $P - C_1$ would have positive genus because C_1 does not bound a disk on P, so that the genus of P would be at least two, thereby contradicting the fact that P is a projective plane. On the other hand, if $C_1 \cup C_2$ did not separate P, then by cutting P along C_1 and C_2 we would obtain a connected 2-manifold with at least two boundary components. P would then have either two cross-caps or a handle, which is again a contradiction. Therefore $C_1 \cup C_2$ separates P into two components, whose closures will be denoted by A and B.

Since the genus of P is one, either A or B, say A, must be of genus zero. So A is an annular ring. Then B is of genus one, and so is a projective plane with two holes. But this implies that P is a 2-sphere with one cross-cap and one handle which is a contradiction. C_1 and C_2 must therefore intersect.

If A and B are disjoint homeomorphic closed 2-manifolds imbedded in a compact 3-manifold M, then A and B will be said to be *concentric* if there is a region $R \subset M$ such that $\mathrm{Bd}\, R = A \cup B$ and R is homeomorphic to $A \times I$.

Lemma 14. *Let P and P' be projective planes, with P' imbedded in $P \times I$ and separating $P \times 0$ and $P \times 1$ in $P \times I$. Then P' is concentric with $P \times 0$ and $P \times 1$.*

Proof. Let p be a fixed point of P. We first alter P', by a homeomorphism of $P \times I$ onto itself, to obtain the situation that the new P' intersects the polygonal arc $p \times I$ in a single point.

Let C be a non-nullhomotopic simple closed polygon on P containing the point p. Assuming that P' is in general position with respect to $R = C \times I$, $P' \cap R$ is a finite collection $J_0, J_1, \cdots, J_n$ of mutually disjoint simple closed curves. By virtue of Lemma 13, only one of these, say J_0, is non-nullhomotopic on P'. Each of the others, $J_1, \cdots, J_n$, is also nullhomotopic on R, since otherwise it would be homotopic in R to $C \times 0$, which is not nullhomotopic in $P \times I$. Hence each of $J_1, \cdots, J_n$ can be eliminated by "exchanging" the disks which it bounds on P' and R (as in the removal of the components of type 1 in the proof of Lemma 10), leaving the single component J_0 of $P' \cap R$. J_0 must be non-nullhomotopic on P', since otherwise it could be eliminated also, thereby contradicting the fact that P' separates $P \times 0$ and $P \times 1$ in $P \times I$.

Now J_0 clearly pierces $p \times I$ in an odd number of points, and is therefore non-nullhomotopic in R also. If J_0 intersects $p \times I$ in more than one point, then there is a subarc A of J_0 which intersects $p \times I$ only in its endpoints a and b, with the union of A and the subarc ab of $p \times I$ being nullhomotopic on R. As in the proof of Lemma 9, if the disk E is the regular neighborhood of A in a very fine subdivision of P', then $m = \mathrm{Bd}\, E$ is a simple closed polygon on P' which separates $a \cup b$ and the rest of the points of $J_0 \cap (p \times I)$. Then m bounds a disk E' in $P \times I$ which does not intersect $p \times I$. If $\mathrm{Int}\, E'$ intersects P', then by Lemma 7 each component of $P' \cap (\mathrm{Int}\, E')$ bounds a disk on P', so that all such components can be eliminated by "exchanging" disks on P' and E'. We then eliminate the points a and b of $P' \cap (p \times I)$ by moving the disk E of P' onto E'.

It may therefore be assumed that $P' \cap (p \times I)$ is a single point. If D is now a sufficiently small disk on P containing p in its interior, then $D \times I$ intersects P' in a disk D' spanning $D \times I$. Since the complement of a disk in a projective plane is a Moebius strip, the lemma now follows from Lemma 12.

Throughout this paper we use (sometimes without explicit mention) the fact that, if A is a compact 2-manifold other than S^2 and S is a polyhedral 2-sphere in $A \times I$, then S bounds a 3-cell in $A \times I$. This is justified by the following lemma.

LEMMA 15. *Let A be a compact 2-manifold other than S^2 (with or without boundary), and let S be a polyhedral 2-sphere in $A \times I$. Then S is the boundary of a 3-cell in $A \times I$.*

Proof. It suffices to consider the case in which $S \subset \mathrm{Int}(A \times I)$. If A is orientable, then $A \times I$ can be considered to be piecewise–linearly imbedded in the 3-sphere S^3. By Alexander's Theorem [1] it follows that the closure of each component of $S^3 - S$ is a closed 3-cell. Now S does not separate $\mathrm{Bd}(A \times I)$ in $A \times I$. For if $\mathrm{Bd}\, A \neq \emptyset$ then $\mathrm{Bd}(A \times I)$ is connected, while if A is a closed orientable 2-manifold of positive genus, then this fact follows from Lemma 5. Thus $\mathrm{Bd}(A \times I)$ must be contained in a single component of $S^3 - S$. Then the closure of the other component is a 3-cell in $A \times I$ which is bounded by S.

Now consider a compact nonorientable 2-manifold A of genus $p > 0$. Let J be a center-line of a Moebius strip in A. By general position it may be assumed that each component of $S \cap (J \times I)$ is a simple closed polygon. Let F be such a component which bounds a subdisk D of S such that $(\mathrm{Int}\, D) \cap (J \times I) = \emptyset$. Since $J \times 0$ is non-nullhomotopic in $A \times I$, F must be nullhomotopic in $J \times I$. So F bounds a subdisk D' of $J \times I$, and it may be assumed (after a preliminary deformation, if necessary) that $D' \subset J' \times I$, where J' is a subarc of J. Now cutting A along J yields a 2-manifold B of genus $p-1$. By induction, the 2-sphere $D \cup D'$ bounds a 3-cell in $B \times I$ and consequently in $A \times I$. The component F of $S \cap (J \times I)$ can then be eliminated by deforming D across C onto D' and then off $J \times I$. Since this process reduces by at least one the number of components of $S \cap (J \times I)$, it may now be assumed that $S \cap (J \times I) = \emptyset$. But then another application of the inductive hypothesis yields a 3-cell in $A \times I$ whose boundary is S.

3. The main concentricity theorems.

THEOREM 1. *Let A and B be homeomorphic closed 2-manifolds, with B tamely imbedded in $\mathrm{Int}(A \times I)$ in such a way that it separates $A \times 0$ and $A \times 1$ in $A \times I$. Then B is concentric with each of $A \times 0$ and $A \times 1$.*

Proof. Since B is tame, it may be assumed that $A \times I$ is triangulated in such a way that B is polyhedral, and such that $K \times I$ is polyhedral if $K \subset A$ is polyhedral.

Suppose first that A and B are orientable of positive genus (since the genus zero case is well known). If A and B are tori, then it may be shown exactly as in the proof of Lemma 11 that B is concentric with $A \times 0$ and $A \times 1$. This was also proved in [10].

We therefore assume that the genus n of A and B is at least two. As in the proof of Lemma 5, let each of $T_1, \cdots, T_n$ be a torus with one hole and K a 2-sphere with n holes, such that $A = K \cup T_1 \cup \cdots \cup T_n$, with each $K \cap T_i$ being the boundary curve J_i of T_i, and $\text{Bd}\, K = \bigcup_1^n J_i$. We define in several steps a homeomorphism of $A \times I$ onto itself, which is the identity on $\text{Bd}(A \times I)$, and such that the image of B intersects the annular ring $R_i = J_i \times I$ in the simple closed curve $J_i \times \frac{1}{2}$ for each $i = 1, \cdots, n$. The theorem will then follow from Lemmas 10 and 11.

Step 1. First B is moved into general position with respect to each of the R_i. Then each $B \cap R_i$ consists of a finite number of mutually disjoint simple closed curves. These are of two types—those which are nullhomotopic in R_i and those which are not nullhomotopic in R_i. Since B separates $A \times 0$ and $A \times 1$ in $A \times I$, $B \cap (T_i \times I)$ must separate $T_i \times 0$ and $T_i \times 1$ in $T_i \times I$, for each i. Since each component of $B \cap (T_i \times I)$ is a compact orientable 2-manifold whose interior lies in $\text{Int}\,(T_i \times I)$ and whose boundary is contained in $\text{Int}(J_i \times I)$, it follows from Lemma 2 that at least one component of $B \cap (T_i \times I)$ is of positive genus. For each i, denote by S_i this positive genus component. Then, since the genus of B is at least as great as the sum of the genera of all the components of all of the $B \cap (T_i \times I)$, it follows that each of the S_i is of genus exactly 1, while each of the other components of $B \cap (T_i \times I)$ is of genus zero.

Step 2. The nullhomotopic components of $B \cap R_i$ are now eliminated. Considering a fixed R_i, let C be a minimal nullhomotopic simple closed curve in $B \cap R_i$, i.e., one which bounds a disk D on R_i whose interior does not intersect B. Then Lemma 8 implies that C bounds a disk D' on B. The component C of $B \cap R_i$ can now be eliminated by a homeomorphism of $A \times I$ onto itself, fixed on $\text{Bd}\,(A \times I)$, which first moves D' onto D and then deforms D slightly off of R_i. Consequently a finite sequence of steps of this kind eliminates all of the nullhomotopic components of all of the $B \cap R_i$. We may therefore assume that B intersects each R_i only in simple closed curves which are non-nullhomotopic in R_i.

Step 3. In this step it is shown that each of the genus zero components of $B \cap (T_i \times I)$ is an annular ring. To the contrary, suppose that W is a component of $B \cap (T_i \times I)$ of genus zero but with more than two boundary components $r_1, \cdots, r_k$. Sew a disk D onto T_i to obtain a torus T_i^*, and think of $T_i^* \times I$ as imbedded in E^3 with $D \times I$ a portion of a very thin vertical circular cylinder Z, on which $r_1, \cdots, r_k$ are horizontal circles. Denote by D_j the horizontal spanning disk of Z whose boundary is $r_j, j = 1, \cdots, k$. It now follows from Lemma 9 that on the polyhedral 2-sphere $S = W \cup D_1 \cup \cdots \cup D_k$ there is a simple closed curve $m \subset S - \bigcup_1^k D_j$ which separates $\bigcup_1^k D_j$ on S and bounds a disk $E \subset (T_i^* \times I) - \text{Int} Z = T_i \times I$, with $B \cap \text{Int}\, E$ being a finite collection of mutually disjoint simple closed curves.

It follows from Lemma 8 that m is the boundary of a disk F on B. But since m separates $\bigcup_1^k r_j$ in W, F then contains at least oneof the r_j, say r_1. So r_1 bounds a disk on B. But this is impossible, because r_1 is homotopic to $J_i \times 0$ in $A \times I$, while $J_i \times 0$ is non-nullhomotopic in $A \times I$. This contradiction proves that each component of genus zero of $B \cap (T_i \times I)$ is an annular ring.

Step 4. These annular rings of $B \cap (T_i \times I)$ are now eliminated. Let R be a component of $B \cap (T_i \times I)$ which is an annular ring whose two boundary components bound an annular ring $R' \subset R_i$ whose interior does not intersect B. This component R of $B \cap (T_i \times I)$ can now be removed by a homeomorphism of $A \times I$ onto itself, fixed on $\mathrm{Bd}(A \times I)$, which first moves R onto R' and then deforms R' slightly off of R_i. This can be done because it is shown in the next paragraph that the torus $R \cup R'$ is the boundary of a solid torus in $T_i \times I$.

In order to see that $R \cup R'$ bounds a solid torus in $T_i \times I$, think of T_i as a subset of the torus $T_i = T_i^* \cup D$, where D is a disk such that $D \cap T_i^* = \mathrm{Bd}\, D = \mathrm{Bd}\, T_i^*$, with $T_i^* \times I$ imbedded piecewise-linearly in S^3. Denote by r_1 and r_2 the boundary components of R. Choose a point $p \in \mathrm{Int}\, D$, and let D_1 and D_2 be disjoint polyhedral disks in $D \times I$ bounded by r_1 and r_2 respectively, each of which is pierced at a single point by the polygonal arc $X = p \times I$. If V is the closure of that component of $S^3 - (R \cup R')$ which is contained in $T_i \times I$, then r_1 and r_2 are non-nullhomotopic curves on the torus $R \cup R'$ which bound the disks D_1 and D_2 in $V' = \mathrm{Cl}(S^3 - V)$, so it follows that V' is a solid torus, of which D_1 and D_2 are meridianal disks [14]. Now let m be a non-nullhomotopic simple closed polygon on T_i^*, which contains the point p and intersects the disk D in a single spanning arc. Assuming that the 2-sphere $S = R \cup D_1 \cup D_2$ is in general position with respect to the annular ring $R^* = m \times I$, $S \cap R^*$ is a finite collection of mutually disjoint simple closed curves in $\mathrm{Int}\, R^*$. Each of these is nullhomotopic in R^* because the boundary curves of R^* are non-nullhomotopic in $T_i^* \times I$. Since S intersects $D \times I$ only in the disks D_1 and D_2, which we may assume to be parameter disks of $D \times I$, only one of the components of $S \cap R^*$ intersects $D \times I$. There is therefore an arc Y in R^*, joining the boundary components $m \times 0$ and $m \times 1$ of R^* with $\mathrm{Int}\, Y \subset \mathrm{Int}\, R^*$, such that Y intersects neither S nor $D \times I$. If E is the closure of one of the components of $R^* - (X \cup Y)$, and F is a parameter disk of $D \times I$ lying between D_1 and D_2, then $e = \mathrm{Bd}\, E$ is an unknotted simple closed curve in $\mathrm{Int}\, V'$ which pierces the meridianal disk F of V' at a single point. It now follows from [14, p. 171] that the center-line of the solid torus V' is unknotted, so that $V = \mathrm{Cl}(S^3 - V')$ is also a solid torus.

Consequently the component R of $B \cap (T_i \times I)$ can be eliminated as indicated above. It may therefore be assumed that, for each i, $B \cap (T_i \times I)$ consists of the single component S_i which is a torus with holes.

Step 5. We now show that there is a component L of $B \cap (K \times I)$ which is a sphere with n holes and boundary components $s_1, \cdots, s_n$ such that, for each i, s_i is a non-nullhomotopic curve on R_i. First, since $B \cap (K \times I)$ must separate

$K \times 0$ and $K \times 1$ in $K \times I$, it is easily seen using the Phragmen-Brouwer property of S^3 that some one component L of $B \cap (K \times I)$ must separate $K \times 0$ and $K \times 1$ in $K \times I$. Each component of $B \cap (K \times I)$, including L, is a 2-sphere with holes because the sum of the genera of the S_i is the genus n of B. Furthermore each R_i must clearly contain a boundary component s_i of L which is non-nullhomotopic in R_i. If some R_i contained another boundary component s of L, then, by sewing all the S_i to L along the curves $s_1, \cdots, s_n, s$, we would obtain a compact 2-manifold of genus $n+1$ in B, which is impossible. Therefore $\mathrm{Bd}\, L = \bigcup_1^n s_i$.

Step 6. It is finally shown that L is the only component of $B \cap (K \times I)$. If there is another one, denote it by L'. Since no non-nullhomotopic simple closed curve in any R_i is nullhomotopic in $K \times I$, L' must have at least two boundary components. Since the genus of $L \cup S_1 \cup \cdots \cup S_n$ is n, it follows that the genus of $L \cup L' \cup S_1 \cup \cdots \cup S_n \subset B$ is at least $n+1$, which is impossible. Thus $B \cap (K \times I) = L$. Therefore each S_i is a torus with exactly one hole, intersecting L in its boundary curve s_i. Finally, we may move s_i onto $J_i \times \frac{1}{2}$.

Thus we have constructed a homeomorphism of $A \times I$ onto itself, the identity on $\mathrm{Bd}(A \times I)$, such that the image of B intersects each R_i in the single simple closed curve $J_i \times \frac{1}{2}$. It now follows from Lemmas 10 and 11 that there is a homeomorphism h of $A \times I$ onto itself such that $h \mid \mathrm{Bd}(A \times I) = 1$ and $h(B) = A \times \frac{1}{2}$, thereby concluding the proof of Theorem 1 in the orientable case.

The proof in case A and B are nonorientable closed 2-manifolds of genus $n > 0$ follows the same plan. By virtue of Lemma 14 it may be assumed that $n > 1$. Then A can be expressed as the union of $K, M_1, \cdots, M_n$, where K is a 2-sphere with n holes and each M_i is a Moebius strip, with each $K \cap M_i$ being the boundary curve J_i of M_i. We will define in steps $1'-6'$ below (analogous to steps $1-6$ above) a homeomorphism of $A \times I$ onto itself, which is the identity on $\mathrm{Bd}(A \times I)$, and such that the image of B intersects the annulus $R_i = J_i \times I$ in the simple closed curve $J_i \times \frac{1}{2}$, $i = 1, \cdots, n$. The nonorientable case will then follow from Lemmas 10 and 12. Details will be omitted wherever their inclusion would be unduly repetitious.

Step $1'$. By general position we may assume that each $B \cap R_i$ consists of a finite number of mutually disjoint simple closed polygons. Just as in step 1 above, we show (using Lemma 3 instead of Lemma 2) that exactly one component S_i of $B \cap (M_i \times I)$ is nonorientable of genus one, with the other components being of genus zero.

Step $2'$. The nullhomotopic components of $B \cap R_i$ are eliminated just as in step 2 above. The process of "exchanging disks" is permissible because of Lemma 15.

Step $3'$. It is shown now that each of the remaining genus zero components of $B \cap (M_i \times I)$ is an annulus. To the contrary suppose that W is a genus zero component of $B \cap (M_i \times I)$ having more than two boundary components $r_1, \cdots, r_k$. Sew a disk D onto M_i to obtain a projective plane P_i, and consider $P_i \times I$, with each r_j of the form $(\mathrm{Bd}\, M_i) \times t_j$, $0 < t_j < 1$. Then denote by D_j the disk $D \times t_j$, $j = 1, \cdots, k$. Let p be a fixed interior point of D, and let $L = p \times I$. Then the 2-sphere $S = W \cup D_1 \cup \cdots \cup D_k$ intersects L in the k points $p \times t_j$, $j = 1, \cdots, k$. It now follows from Lemma 9 that S contains a simple closed polygon m' which separates the points $\{p \times t_j\}_{j=1}^{k}$ on S and bounds a disk E' which misses L. By a simple radial deformation away from L we obtain a simple closed polygon $m \subset S - \bigcup_1^k D_j$ which separates $\bigcup_1^k D_j$ on S and bounds a disk $E \subset M_i \times I$, with $B \cap \mathrm{Int}\, E$ being a finite number of mutually disjoint polygons. A contradiction now follows just as in step 3 above.

Step 4′. These annular rings of $B \cap (M_i \times I)$ are now removed. Let R be an annulus of $B \cap (M_i \times I)$ whose two boundary components bound an annulus $R' \subset R_i$ such that $B \cap \mathrm{Int}\, R' = \varnothing$. In order to show that the component R of $B \cap (M_i \times I)$ can be removed as in step 4 above, it suffices to show that the torus $R \cup R'$ bounds a solid torus in $M_i \times I$.

Denote by C_i a center-line of the Moebius strip M_i. By general position it may be assumed that each component of $R \cap (C_i \times I) = \mathrm{Int}\, R \cap \mathrm{Int}(C_i \times I)$ is a simple closed polygon. Since every polyhedral 2-sphere in $M_i \times I$ bounds a 3-cell (Lemma 15), in order to show that the components of $R \cap (C_i \times I)$ can be removed by a process of "exchanging disks," it suffices to show that every component X of $R \cap (C_i \times I)$ is nullhomotopic in both R and $C_i \times I$. First note that X is nullhomotopic in R if and only if it is nullhomotopic in $C_i \times I$. This follows from the fact that a simple closed curve is nullhomotopic in R (or $C_i \times I$) if and only if it is nullhomotopic in $M_i \times I$, because $J_i \times 0$ and $C_i \times 0$ are both non-nullhomotopic in $M_i \times I$. It remains to show that X cannot be non-nullhomotopic in both R and $C_i \times I$. But if it were, then it would be homotopic (in $M_i \times I$) to both $J_i \times 0$ and $C_i \times 0$, which are not homotopic in $M_i \times I$. Thus every component of $R \cap (C_i \times I)$ bounds a disk in both R and $C_i \times I$.

It may therefore be assumed that $R \cap (C_i \times I) = \varnothing$. Now cut $M_i \times I$ along the annulus $C_i \times I$ to obtain a solid torus T. It suffices to show that $R \cup R'$ bounds a solid torus in T. Let D be a meridianal disk of T which intersects the annulus R' in a spanning arc G' with end points a and b. By general position it may be assumed that each component of $D \cap \mathrm{Int}\, R$ is of one of the following three types: (1) a simple closed polygon, (2) an open arc whose closure is a simple closed curve, (3) an open arc whose closure is a closed arc with endpoints a and b. By precisely the elimination process used in the proof of Lemma 10, all but one of these components can be removed, leaving a single spanning arc G of R with endpoints a and b. Similarly, if E is a second meridianal disk of T, disjoint with D and intersecting R' in a spanning arc H' with endpoints c and d, then the unnecessary components of $E \cap R$ can be removed, leaving a single spanning arc H of R with endpoints c and d. Denote by D' and E' the subdisks of D and E bounded by $G \cup G'$ and $H \cup H'$ respectively, and by S_1 and S_2 the annuli into which $D \cup E$ separates the torus $R \cup R'$. Then the 2-spheres $S_1 \cup D' \cup E'$ and

$S_2 \cup D' \cup E'$ bound 3-cells C_1 and C_2 respectively in T. Then $C_1 \cup C_2$ is a solid torus in T which is bounded by the torus $R \cup R'$.

It may therefore be assumed that, for each i, $B \cap (M_i \times I)$ consists of the single component S_i, which is a projective plane with holes.

Step 5'. The argument of step 5 above now applies to show that there is a component L of $B \cap (K \times I)$ which is a 2-sphere with n holes and boundary components $s_1, \cdots, s_n$ such that, for each i, s_i is non-nullhomotopic in R_i.

Step 6'. Finally the argument of step 6 above applies to show that L is the only component of $B \cap (K \times I)$. Hence each S_i is a Moebius strip which intersects L in its boundary curve s_i. So Lemmas 10 and 12 apply to complete the proof of Theorem 1 in the nonorientable case.

THEOREM 2. *Let M_0, M_1, and M be compact* 3-*manifolds with homeomorphic nonempty boundaries, with M_0 and M tamely imbedded in* Int$M$ *and* IntM_1 *respectively. If M_0 and M_1 are concentric, then M is concentric with* (*and hence homeomorphic with*) *each of M_0 and M_1.*

Proof. Let $A_1, \cdots, A_p$ be the boundary components of M_0, so that M_1 can be expressed as $M_0 \cup \bigcup_1^p A_i \times I$ with each $A_i \times 0$ identified with A_i. By invariance of domain, Int$M$ is open in M_1 and of course $M_1 - M$ is open in M_1. Thus BdM separates BdM_0 and BdM_1 in M_1. Hence each $A_i \times I$ must contain a boundary component of M. Because BdM and BdM_0 are homeomorphic, and in particular have the same number of components, each of the $A_i \times I$ must contain exactly one component B_i of BdM, with B_i separating $A_i \times 0$ and $A_i \times 1$ in $A_i \times I$. By Lemma 5, A_i and B_i are either both orientable or both nonorientable with genus $B_i \geqq$ genus A_i. Since BdM_0 and BdM are homeomorphic, and in particular their composite genera are equal, it follows that A_i and B_i have the same genus and are therefore homeomorphic. Theorem 1 now applies to show that, for each i, B_i is concentric with $A_i \times 0$ and $A_i \times 1$. It follows easily that M is concentric with M_0 and M_1. Since the boundary of a compact 3-manifold with boundary is collared ([7] or [13, Lemma 1]), any two concentric 3-manifolds are necessarily homeomorphic.

4. **Applications.** In this section we give several applications of Theorems 1 and 2.

THEOREM 3. *If M and N are two compact* 3-*manifolds whose interiors are homeomorphic, then M and N are homeomorphic.*

Proof. If M and N are closed there is nothing to prove. Otherwise, since the boundary of a manifold with boundary is collared [7], there is a compact 3-manifold $M' \subset$ Int$M$ such that M' and M are concentric. For the same reason the interior of N can be expressed as the union of a sequence $\{N_i\}_1^\infty$ of compact 3-manifolds, each homeomorphic with N, such that, for each i, $N_i \subset$ IntN_{i+1} and BdN_i is bicollared in Int$N$.

Now let h be a homeomorphism of $\operatorname{Int} N$ onto $\operatorname{Int} M$. Then, for each i, $\operatorname{Bd} h(N_i)$ is bicollared in $\operatorname{Int} M$, so that each $h(N_i)$ is tamely imbedded in $\operatorname{Int} M$ [5]. By compactness there is an integer k such that $M' \subset \operatorname{Int} h(N_k) \subset h(N_k) \subset \operatorname{Int} M$, and we may assume that $h(N_k)$ is polyhedral in M. If it were known that $\operatorname{Bd} M$ and $\operatorname{Bd} h(N_k)$ were homeomorphic, Theorem 2 would then imply that $h(N_k)$ is concentric with M' and M.

Just as in the proof of Theorem 2, we see that $\operatorname{Bd} h(N_k)$ separates $\operatorname{Bd} M'$ and $\operatorname{Bd} M$ in M, so that $\operatorname{Bd} h(N_k)$ and hence N has at least as many boundary components as does M. Since by symmetry the reverse inequality also holds, $\operatorname{Bd} M$ and $\operatorname{Bd} N$ have the same number of components. Hence, if A_j and A_j' are corresponding components of $\operatorname{Bd} M$ and $\operatorname{Bd} M'$ respectively, denote by B_j the unique component of $\operatorname{Bd} h(N_k)$ separating A_j and A_j' in M, $j = 1, \cdots, r$. Then, by Lemma 5, it is true for each j that A_j and B_j are either both orientable or both nonorientable, and the genus of B_j is at least as great as the genus of A_j. But again by symmetry the reverse inequality also holds, so that A_j and B_j are homeomorphic for each $j = 1, \cdots, r$. Hence $\operatorname{Bd} M$ and $\operatorname{Bd} h(N_k)$ are homeomorphic.

Thus Theorem 2 applies to show that M and $h(N_k)$ are concentric, so that M and N are homeomorphic, $h(N_k)$ being homeomorphic with N.

In [9] and [10] the author showed that the 3-sphere S^3 does not contain an uncountable collection of mutually disjoint tori, no two of which are concentric. The proof given there relied heavily upon the genus one case of Theorem 1, and therefore did not then extend to the case of closed 2-manifolds of arbitrary genus. However, shortly thereafter Gillman [11] was able to circumvent this difficulty by giving a different proof based on a result of Bing [5]. Since Theorem 1 is now available, the general result is stated below.

THEOREM 4. *If $\mathcal{G}$ is an uncountable collection of mutually disjoint two-sided closed 2-manifolds in the compact 3-manifold M, then $\mathcal{G}$ contains a pair of concentric elements.*

Proof. Since there is only a countable number of topologically distinct closed 2-manifolds, it may be assumed that each two elements of $\mathcal{G}$ are homeomorphic. By virtue of a result of Bing [4], it may also be assumed that each element of $\mathcal{G}$ is tamely imbedded in M. Then, just as in the proof of Theorem 13 of [10], it can be shown that $\mathcal{G}$ contains an element A such that, if C is a collar of A in M (i.e., $C = h(A \times I)$ with $h(x,0) = x$ if $x \in A$), then $\mathcal{G}$ contains an element B which separates the boundary components of C. Theorem 1 then implies that A and B are concentric.

M. Brown and B. Cassler [8, pp. 92–94] have shown that, if M is a compact n-manifold with nonempty boundary B, then there is a continuous map ϕ of $B \times I$ onto M such that (a) $\phi(x,1) = x$ if $x \in B$, (b) $\phi \mid B \times (0,1]$ is a homeomorphism onto $M - \phi(B \times 0)$, and (c) $\phi(B \times 0)$ is of dimension less than n. If K is a subset of M such that such a map exists with $\phi(B \times 0) = K$, then K will be called a *spine* of M and it will be said that ϕ *displays* K as a spine of M. If M is triangulated, then a polyhedral spine of M can be obtained in the standard way by "pushing in" exposed $(n-1)$-simplexes until no n-simplexes remain. However, in the following two results, spines are alowed to be nonpolyhedral.

LEMMA 16. *Let M be a compact 3-manifold with boundary B and spine K. If h is a homeomorphism of M into M such that $h(M)$ is tame in* $\operatorname{Int} M$ *and $h(K)=K$, then M and $h(M)$ are concentric.*

Proof. Let ϕ be a map of $B \times I$ onto M which displays K as a spine of M. By invariance of domain, $h(\operatorname{Int} M)$ is a neighborhood of $h(K)=K$. Hence $t>0$ can be chosen so small that $\phi(B \times [0,t]) \subset \operatorname{Int} h(M)$. Since $\phi(B \times [0,t])$ is clearly concentric with M, it follows from Theorem 2 that M and $h(M)$ are concentric.

If $\{C_n\}_1^\infty$ is a sequence of 3-cells, each interior to its successor, then $\bigcup C_n$ is an open 3-cell [6]. Similarly, if $\{T_n\}_1^\infty$ is a sequence of genus 1 solid tori, each interior to its successor, with all the T_n sharing a common center-line, then it can be shown that $\bigcup T_n$ is homeomorphic to $S^1 \times E^2$. Theorem 5 below is a generalization of these two facts, since a point is a spine of a 3-cell and a spine of a solid torus is its center-line.

THEOREM 5. *Let M be a compact 3-manifold with boundary B and spine K. If the open 3-manifold X is the union of a sequence $\{h_n(M)\}_1^\infty$ of homeomorphic images of M such that, for each n, $h_n(M) \subset \operatorname{Int} h_{n+1}(M)$ and $h_n(K)=h_{n+1}(K)$, then X is homeomorphic to* $\operatorname{Int} M$.

Proof. Since the boundary of a compact manifold is collared, there is for each $n>1$ homeomorphism r_n of $h_n(M)$ into itself such that $r_n \mid h_{n-1}(M)$ is the identity and $r_n h_n(\operatorname{Bd} M)$ is bicollared in $\operatorname{Int} h_n(M)$. Consequently it may be assumed that, for each n, $h_n(M)$ is tamely imbedded in $h_{n+1}(M)$.

If ϕ is a map of $B \times I$ onto M displaying K as a spine of M, define $\phi_n(x,t) = h_n(\phi(h_n^{-1}(x),t))$ for $x \in \operatorname{Bd} h_n(M) = h_n(B)$ and $t \in I$. Then ϕ_n is a map of $h_n(B) \times I$ onto $h_n(M)$ displays $h_n(K) = h_1(K)$ as a spine of $h_n(M)$. Therefore $h_n h_{n+1}^{-1} : h_{n+1}(M) \to h_n(M)$ is a homeomorphism of $h_{n+1}(M)$ into itself which satisfies the hypotheses of Lemma 16. Consequently $h_n(M)$ and $h_{n+1}(M)$ are concentric for each n.

Now $\operatorname{Int} M$ can be expressed as an increasing union $\{M_n\}_1^\infty$ of copies of M such that M_n and M_{n+1} are concentric for each n. Let g_1 be an arbitrary homeomorphism of $h_1(M)$ onto M_1. Having defined a homeomorphism g_n of $h_n(M)$ onto M_n agreeing with g_{n-1} on $h_{n-1}(M)$, use the fact that $h_{n+1}(M) - \operatorname{Int} h_n(M)$ and $M_{n+1} - \operatorname{Int} M_n$ are both homeomorphic with $B \times I$ to extend g_n to a homeomorphism g_{n+1} of $h_{n+1}(M)$ onto M_{n+1}. Then $g = \lim g_n$ is a homeomorphism of X onto $\operatorname{Int} M$.

THEOREM 6. *Let the compact Hausdorff space X be the union of two open* *subsets* $U_i = M_i \times [0,1)$, $i = 1,2$, *with* $M_1 \times 0$ *and* $M_2 \times 0$ *disjoint, where* M_1 *and* M_2 *are closed* 2-*manifolds. Then* X *is homeomorphic with* $M_i \times [0,1]$, $i = 1,2$.

Proof. Since X is compact and U_1 is open, $\operatorname{Bd} U_1$ is nonempty and is contained in U_2. Since $\{(M_1 \times [1 - 1/n, 1)) \cup \operatorname{Bd} U_1\}_1^\infty$ is a monotone decreasing sequence of compact sets intersecting in $\operatorname{Bd} U_1$, an $\varepsilon_1 \in (0,1)$ can be chosen such that $M_1 \times [\varepsilon_1, 1) \subset U_2$. Similarly $\varepsilon_2 \in (0,1)$ can be chosen such that $M_2 \times [\varepsilon_2, 1) \subset U_1$ and $M_1 \times [\varepsilon_1, 1) \subset M_2 \times [0, \varepsilon_2)$.

Then $M_2 \times \varepsilon_2$ is contained in the open subset $M_1 \times [0, \varepsilon_1)$ of X, while $M_2 \times 0 \subset X - U_1 \subset X - (M_1 \times [0, \varepsilon_1])$ because $M_1 \times 0$ and $M_2 \times 0$ are disjoint. Hence $M_1 \times \varepsilon_1$ separates $M_2 \times 0$ and $M_2 \times \varepsilon_2$ in X and therefore in $M_2 \times [0, \varepsilon_2]$. Since $M_1 \times \varepsilon_1$ is bicollared and hence tame in $M_2 \times [0, \varepsilon_2]$, Lemma 5 implies that M_1 and M_2 are either both orientable or both nonorientable and that the genus of M_1 is equal to or greater than the genus of M_2. But symmetrically the genus of M_2 is at least as large as the genus of M_1. Consequently M_1 and M_2 are homeomorphic.

Theorem 1 now applies to show that the closures of the components A and B of $M_2 \times [0, \varepsilon_2] - M_1 \times \varepsilon_1$ containing $M_2 \times 0$ and $M_2 \times \varepsilon_2$ respectively are both homeomorphic to $M_1 \times I$.

Now $M_2 \times \varepsilon_2 \subset M_1 \times [0, \varepsilon_1)$ and $B \cup (M_2 \times [\varepsilon_2, 1))$ is a connected set not intersecting $M_1 \times \varepsilon_1$, so it follows that $U_2 - \operatorname{Cl} A \subset M_1 \times [0, \varepsilon_1)$. Since $M_1 \times (\varepsilon_1, 1)$ is a connected subset of U_2 not intersecting $M_1 \times \varepsilon_1$, it follows that $M_1 \times (\varepsilon_1, 1) \subset A$. Therefore $X = (M_1 \times [0, \varepsilon_1]) \cup \operatorname{Cl} A$. Since A is a connected subset of $X - (M_1 \times \varepsilon_1)$ containing $M_2 \times 0 \subset X - (M_1 \times [0, \varepsilon_1])$, $\operatorname{Cl} A$ and $M_1 \times [0, \varepsilon_1]$ intersect in $M_1 \times \varepsilon_1$. Because $\operatorname{Cl} A$ is homeomorphic to $M_1 \times [\varepsilon_1, 1]$, it now follows that X is homeomorphic to $M_1 \times [0,1]$.

We conclude with an application to the imbedding of finite graphs in 3-manifolds.

THEOREM 7. *Let G be a finite graph tamely imbedded in the orientable closed* 3-*manifold M, and denote by p the* 1-*dimensional Betti number of G. Suppose that $\{H_n\}_1^\infty$ is a sequence of tame compact* 3-*manifolds intersecting in G such that, for each n, $H_{n+1} \subset \operatorname{Int} H_n$ and $\operatorname{Bd} H_n$ is a closed orientable* 2-*manifold of genus p. Then there is an N such that H_n and H_{n+1} are concentric solid tori of genus p if $n \geqq N$.*

Proof. It may be assumed without loss that G is polyhedral. Then the normal regular neighborhood R of K in a given triangulation of M is a solid torus of genus p. Choose N such that $H_n \subset \operatorname{Int} R$ for all $n \geqq N$. Then, given $n \geqq N$, let M be subdivided so finely that the normal regular neighborhood R' of G in the subdivision lies interior to H_{n+1}. Now each of R and R' has spine K, and there is a piecewise-linear homeomorphism of R onto R' which leaves K invariant [16].

Therefore Lemma 15 implies that R and R' are concentric. Since $R' \subset \operatorname{Int} H_{n+1} \subset H_{n+1} \subset \operatorname{Int} R$, Theorem 2 shows that H_{n+1} is a solid torus of genus p concentric with R. Since $H_{n+1} \subset \operatorname{Int} H_n \subset H_n \subset R$, a second application of Theorem 2 now shows that H_n is also a solid torus of genus p and is concentric with H_{n+1}.

Added in proof. Since this paper was written, R. L. Finney has announced an independent proof of Theorem 1 (*A condition for regularity in local rings*, Abstract 64T–17, Notices Amer. Math. Soc. **11** (1964), 131), and he has informed the author that E. M. Brown has also proved this theorem.

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